# Problem Set 2

**Instructions**

- Submit individually through the course page on **Campus Virtual**.
- Upload a single PDF with your answers to the analytical parts, together with your code for the numerical parts and the requested figures, if applicable. The preferred language is MATLAB.

In Exercises 1 and 2, a household has preferences $\sum_{t\geq 0}\beta^t u(c_t)$ (in expectation, when income is random) and can only trade a one-period risk-free bond at price $q < 1$, taken as given and constant. Its budget constraint is

$$c_t + q\, a_{t+1} = y_t + a_t,$$

where $a_t$ is the face value of bonds carried into period $t$, $y_t$ is income, and $a_0$ is given.

## Exercise 1: Impatience and the borrowing limit

Suppose income is constant, $y_t = y > 0$, the household is impatient, $\beta < q$, and it faces the borrowing limit $a_{t+1} \geq \underline{a}$ for all $t$. Assume $u$ is strictly increasing, strictly concave and satisfies the Inada conditions, and that $a_0 > \underline{a}$.

**(a)** Suppose $\underline{a} > -y/(1-q)$. Show that the borrowing limit must bind in finite time. *Hint: what does the Euler equation imply for consumption while the limit is slack? Use the intertemporal budget constraint.*

**(b)** Show that once the limit binds, it binds forever, and that consumption is then constant at $c = y + (1-q)\,\underline{a}$. *Hint: guess the continuation plan $a_{t+1} = \underline{a}$ for all remaining $t$, and check that it satisfies all first-order conditions, complementary slackness and the transversality condition. Why are these conditions sufficient here?*

**(c)** Now let $\underline{a} = -y/(1-q)$ and $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$. Solve for the optimal paths $\{c_t\}$ and $\{a_t\}$ in closed form. Show that the borrowing limit never binds, that $\{c_t\} \to 0$, and that $\{a_t\} \to \underline{a}$.

**(d)** Why does the result in (c) not contradict (a)? What is special about the limit $\underline{a} = -y/(1-q)$?

## Exercise 2: How persistence shapes the response of consumption

Consider the quadratic benchmark from class: $u(c) = -\frac{1}{2}(\bar{c} - c)^2$, $\beta = q$, no borrowing limit, and the no-Ponzi condition $\lim_{T\to\infty} \mathbb{E}_t[q^T a_{t+T}] \geq 0$. Recall that the optimal consumption rule

implies

$$\Delta c_t = (1-q)\sum_{s=0}^{\infty} q^s \left(\mathbb{E}_t - \mathbb{E}_{t-1}\right) y_{t+s}.$$

In all cases below, the shocks $\varepsilon_t$ and $\nu_t$ are i.i.d., have mean zero and constant variances $\sigma_\varepsilon^2$ and $\sigma_\nu^2$, and are independent of each other.

**(a) A persistent and a permanent component.** Let

$$y_t = \rho\, y_{t-1} + (1-\rho)\,\eta_t + \varepsilon_t, \qquad \eta_t = \eta_{t-1} + \nu_t, \qquad 0 \le \rho < 1.$$

Find $\Delta c_t$ as a function of $\varepsilon_t$ and $\nu_t$, and compute the marginal propensity to consume out of each shock. What do these become for $\rho = 0$, the permanent-plus-transitory case from class? What do they approach as $\rho \to 1$? Discuss: what determines how much consumption responds to a shock?

**(b) Measuring income risk.** Set $\rho = 0$ in (a), so that $y_t = \eta_t + \varepsilon_t$. Suppose you observe a long panel of consumption and income for many households, each following this process with independent shocks. Assume the interest rate is small, so that $q \approx 1$. Show how to identify $\sigma_\nu^2$ and $\sigma_\varepsilon^2$ from the covariances of $\Delta c_t$ and $\Delta y_t$ at different leads and lags.

**(c) Persistent income growth.** Now let income *growth* be persistent: $\Delta y_t = \rho\, \Delta y_{t-1} + \varepsilon_t$, with $0 < \rho < 1$. Find $\Delta c_t$ as a function of $\varepsilon_t$, and show that $\mathrm{Var}_{t-1}(\Delta c_t) > \sigma_\varepsilon^2$: consumption growth is more volatile than income innovations. Explain why. Aggregate US labor income is often described by this process. Do you think this prediction holds for aggregate consumption in the data? Discuss.

## Exercise 3: Entrepreneurs and financial development

This exercise studies the economy with entrepreneurs from class. There is a unit mass of households with preferences $\mathbb{E}_0 \sum_{t\ge 0} \beta^t u(c_t)$. Each household has labor efficiency $\theta$ and entrepreneurial ability $z$, which follow independent Markov chains. Households save in a one-period bond at price $q = 1/(1+r)$ and cannot borrow, $a' \ge 0$, where $a$ denotes the face value of bonds carried into the period.

At the start of each period, a household with assets $a$ chooses either to work, earning $w\,\theta$, or to run a firm, earning

$$\pi(a,z) = \max_{k \le \phi a,\; l \ge 0} z\, k^{\gamma_k} l^{\gamma_l} - w\, l - R\, k, \qquad \gamma_k \equiv \alpha(1-\nu), \quad \gamma_l \equiv (1-\alpha)(1-\nu), \quad R \equiv r + \delta,$$

with $\nu \in (0,1)$ and $\phi > 0$. Entrepreneurs do not supply labor. The budget constraint is $c + q\,a' = a + \max\{w\,\theta,\ \pi(a,z)\}$. A competitive corporate sector produces with $F(K_c, L_c) = K_c^{\alpha} L_c^{1-\alpha}$, renting capital at $R$ and labor at $w$. As in class, household saving finances all capital, run by

entrepreneurs or by the corporate sector.

**(a)** Write the household's problem in recursive form, in a stationary environment. Define a stationary recursive competitive equilibrium.

**(b)** Consider an entrepreneur for whom the collateral constraint does not bind. Derive output $y^\star(z)$, capital $k^\star(z)$, labor $l^\star(z)$ and profits $\pi^\star(z)$ as functions of $(z, r, w)$. Show that the ratio $k^\star(z)/l^\star(z)$ does not depend on $z$, and that it equals the capital–labor ratio of the corporate sector, $\kappa(r) \equiv K_c/L_c$.

**(c)** Now suppose the constraint binds, $k = \phi a$. Derive $l$ and $\pi(a, z)$. Show that $\pi(\cdot, z)$ is increasing and concave in $a$ up to the threshold $\bar{a}(z) \equiv k^\star(z)/\phi$, and constant above it.

**(d)** Characterize the occupational choice. Show that, whenever $\pi^\star(z) > w\,\theta$, the household's income $\max\{w\,\theta, \pi(a, z)\}$ has a convex kink in $a$. Explain why the value function need not be concave in $a$, and what this implies for solution methods based on the Euler equation, such as the endogenous grid method.

**(e)** Let $L^w$ be the labor supplied by workers and $L^e$ the labor hired by entrepreneurs. Show that the aggregate demand for capital is

$$K = \kappa(r)\, L^w - D, \qquad D \equiv \int_{\text{entrepreneurs}} \big(\kappa(r)\, l - k\big)\, \mathrm{d}\lambda,$$

and that $D \geq 0$, with $D = 0$ if no entrepreneur is constrained. Interpret each term.

**(f)** **Numerical.** The folder `Entrepreneurs_stationary_equilibrium`, among the course codes, contains MATLAB code that computes the stationary equilibrium with the calibration from class; it is organized like the codes for the Aiyagari economy. Read its `README.md`. The script `main.m` computes the equilibrium for $\phi = 1.5$. Two calls do the rest:

- `find_equilibrium(set_parameters(phi))` computes the equilibrium for any $\phi$;
- `excess_supply(r, set_parameters(phi))` solves the households' and firms' problems at a given interest rate, without clearing the capital market.

Compute the equilibrium for $\phi \in \{1, 1.25, 1.5, 2, 3, 4\}$, and plot, as functions of $\phi$: the interest rate; output and TFP, relative to $\phi = 1$; the share of households that are entrepreneurs and the share of capital they run. Report the wealth Gini and the top 1% wealth share.

**(g)** **Why does the interest rate fall?**

(i) Use (e) to predict how a higher $\phi$ affects the demand for capital at a given interest rate. Through which channels? Can you sign the total effect?

(ii) Fix the interest rate at its equilibrium value for $\phi = 1$. Using `excess_supply`, compute each term in (e), and the supply of capital, for $\phi = 1$ and for $\phi = 4$. Which term accounts for the change in the excess supply of capital?

(iii) Households' saving, $q \int a^\star \mathrm{d}\lambda$, is the supply of capital. Split it between entrepreneurs and workers, for $\phi = 1$ and $\phi = 4$. Whose saving rises most? At a given $r$, a worker's income does not depend on $\phi$: why would workers save more? Propose an explanation, and use the model's output to check it.

**(h)** Discuss. What does financial development, a higher $\phi$, do to TFP, output and wealth concentration in this economy? What does your answer to (g) suggest about the link between entrepreneurship and the interest rate?