Constrained Efficiency with Incomplete Markets

Lecture based on "Constrained Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks"
by Dávila, Hong, Krusell and Ríos-Rull, *Econometrica* (2012)

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Too much capital?

- In an Aiyagari economy, households save to self-insure against uninsurable income risk.
- Precautionary saving pushes capital above its complete-markets level, and the interest rate below the discount rate: $r < \rho$.
- Is this capital stock too high, too low, or right, given that insurance markets are missing?
- The answer needs a benchmark: what could a planner do better?

One answer: the first best

- A planner who can transfer goods across households insures everyone fully and equalizes consumption.
- With no risk left, the planner's Euler equation is the representative agent's:

$$1 = \beta\left(1 + f_K(K, L) - \delta\right) \quad \Longleftrightarrow \quad r = \rho,$$

  where $\beta = 1/(1+\rho)$ and f is a constant-returns production function.
- The market economy has $r < \rho$: by this benchmark it **overaccumulates** capital.
- But this planner completes the missing markets. The comparison mixes two things: the missing insurance, and how households use the markets they have.

Constrained efficiency

- Give the planner the same market structure as households:
  - it chooses how much each household saves;
  - it cannot transfer goods across households;
  - factor prices stay competitive: $r = f_K - \delta$, $w = f_L$.
- The equilibrium is **constrained efficient** if this planner cannot raise welfare (Diamond, 1967).
- The only lever is aggregate saving, and it works through factor prices.
- Price changes are transfers: when the wage rises, workers gain what capital owners lose. Why would they ever matter for welfare?

Pecuniary externalities

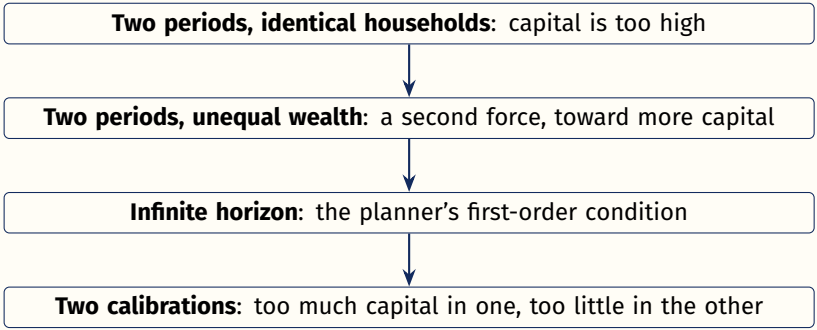
- A small change in prices changes the income of a household with assets a and labor e by $a\,\mathrm{d}r + e\,\mathrm{d}w$.
- With constant returns, the price changes caused by $\mathrm{d}K$ satisfy

$$K\,\mathrm{d}r + L\,\mathrm{d}w = 0,$$

since $Kf_{KK} + Lf_{LK} = 0$. Summed over households, the income changes cancel.

- With complete markets, all marginal utilities are equal, so the welfare changes cancel too.
- With missing markets, marginal utilities differ: a transfer from low to high marginal utility raises welfare. These are **pecuniary externalities**.

Road map



- Is there too much or too little capital? What does it depend on?

Two periods: the mechanism

Ex ante identical households

- A measure one of households live two periods. In period 1 each has ω goods, and consumes or saves them as capital a.
- In period 2 each household rents its capital and its labor e_i to firms, and consumes:

$$c_i = R\,a + w\,e_i, \qquad e_1 < e_2, \quad \Pr(e_1) = \pi.$$

- Capital fully depreciates, so the gross return is $R = f_K(K, L)$; the wage is $w = f_L(K, L)$.
- Shocks are independent across households, so labor supply is constant: $L = \pi e_1 + (1 - \pi)e_2$.
- Capital is the only asset: no insurance against e.

Competitive equilibrium

Competitive equilibrium

Given ω, f and the distribution of e, an equilibrium consists of saving K and prices (R, w) such that (i) K solves

$$\max_a \; u(\omega - a) + \beta\big[\pi u(Ra + we_1) + (1-\pi)u(Ra + we_2)\big];$$

(ii) $R = f_K(K, L)$ and $w = f_L(K, L)$.

- All households are identical in period 1, so they all save $a = K$.
- The Euler equation:

$$u'(\omega - K) = \beta R\left[\pi u'(c_1) + (1-\pi)u'(c_2)\right].$$

Can the market allocation be improved?

- The planner asks every household to save $K + \mathrm{d}K$. Utility changes by

$$\mathrm{d}U = -u'(\omega - K)\,\mathrm{d}K + \beta\,\mathbb{E}\big[u'(c_i)\,(R\,\mathrm{d}K + K\,\mathrm{d}R + e_i\,\mathrm{d}w)\big].$$

- By the Euler equation, the direct effect of saving more is zero:

$$\mathrm{d}U = \beta\,\mathbb{E}\big[u'(c_i)\,(K\,\mathrm{d}R + e_i\,\mathrm{d}w)\big].$$

- Only the price effect remains: the distortion of the saving choice is second order, the price effect first order.

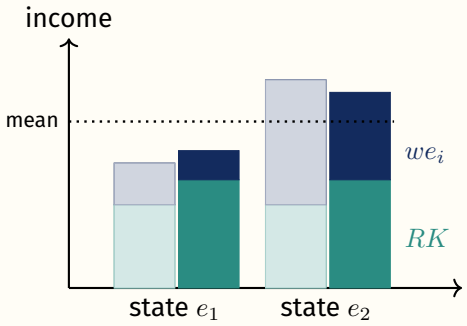
Overaccumulation of capital

- Use $\mathrm{d}R = f_{KK}\,\mathrm{d}K$ and $\mathrm{d}w = f_{LK}\,\mathrm{d}K = -(K/L)f_{KK}\,\mathrm{d}K$:

$$\frac{\mathrm{d}U}{\mathrm{d}K} = \beta f_{KK} K\,\mathbb{E}\Big[u'(c_i)\Big(1 - \frac{e_i}{L}\Big)\Big].$$

- The term $1 - e_i/L$ has mean zero, so the expectation is a covariance between $u'(c_i)$ and $1 - e_i/L$.
- Both are higher in the bad state e_1: the covariance is positive. With $f_{KK} < 0$, $\mathrm{d}U/\mathrm{d}K < 0$.
- Every household gains if all save a little less: the equilibrium has **too much capital**.

Less capital, less risk



- Period-2 income: capital income in green, labor income in blue.
- Lower K: R rises and w falls. Mean income is unchanged, but the risky part, we_i, **shrinks**.

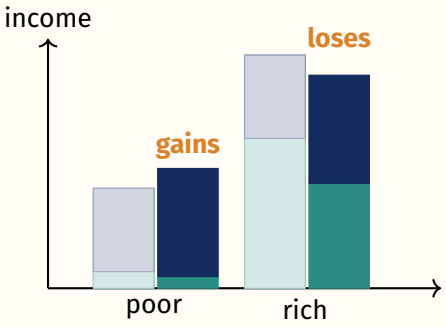
Initial wealth differences

- Now initial wealth ω is distributed as Γ. Saving $a(\omega)$ increases with ω, and $K = \int a(\omega)\,\Gamma(\mathrm{d}\omega)$.
- To isolate the new force, shut down income risk: $e_1 = e_2 = L$.
- The same steps give the effect of more aggregate capital on a household with wealth ω:

$$\frac{\mathrm{d}U(\omega)}{\mathrm{d}K} = \beta f_{KK} K \; u'(c_2(\omega)) \Big(\frac{a(\omega)}{K} - 1 \Big),$$

  where $c_2(\omega) = R\,a(\omega) + wL$.
- The sign depends only on whether the household saves more or less than average.

More capital redistributes toward the wealth-poor



- Period-2 income: the same labor income, in blue, and different capital income, in green.
- Higher K: w rises and R falls. Households with $a(\omega) < K$ gain, those with $a(\omega) > K$ lose.
- Some gain and some lose: no Pareto improvement. Ranking the change needs welfare weights.

A utilitarian planner

- The planner is utilitarian, behind the veil of ignorance.
- For this planner, redistribution from the consumption-rich to the consumption-poor is the insurance that markets do not provide.
- Summing over households, the net effect of more capital:

$$\Delta \equiv \beta f_{KK} K \int u'(c_2(\omega)) \Big(\frac{a(\omega)}{K} - 1 \Big) \, \Gamma(\mathrm{d}\omega).$$

- A marginal-utility-weighted average: the consumption-poor, with $a(\omega) < K$, count most. So $\Delta > 0$: **too little capital**.
- With income risk as well, the bracket becomes $a(\omega)/K - e_i/L$, and Δ adds up both forces: risk, toward less capital, and redistribution, toward more.

Infinite horizon

The Aiyagari economy

- Labor $e \in \{e_1, \ldots, e_I\}$ follows a Markov chain $\pi_{ee'}$, independent across households. Capital depreciates at rate δ.
- The household problem:

$$v(x, e, a) = \max_{c,\, a'} u(c) + \beta \sum_{e'} \pi_{ee'}\, v(x', e', a')$$

$$\text{s.t. } c + a' = a(1 + r(x)) + e\, w(x), \qquad a' \geq \underline{a}, \qquad x' = H(x),$$

  where x is the distribution over (e, a), $r(x) = f_K(K, L) - \delta$, $w(x) = f_L(K, L)$ and $K = \int a \,\mathrm{d}x$.

- The distribution is the aggregate state: future prices depend on it, not on K alone.

Equilibrium and steady state

Recursive competitive equilibrium

Given u, β, f, δ, $\underline{a}$ and $\pi_{ee'}$, an equilibrium consists of a decision rule $a' = h(x, e, a)$ and a law of motion H such that h solves the household problem given H, and $H(x)$ is the distribution that h and $\pi_{ee'}$ generate from x.

- A steady state is a fixed point $\bar{x} = H(\bar{x})$, found as in Aiyagari (1994):
  1. guess r, and set K from $r = f_K(K, L) - \delta$ and $w = f_L(K, L)$;
  2. solve the household problem and find the stationary distribution;
  3. compare the capital it implies with K; update r until they match.
- Precautionary saving gives $r < \rho$ in steady state (Huggett, 1997).

The constrained planner

- At date 0 all households are identical, so the planner is utilitarian, as in the two-period model.
- It chooses saving $y(e,a)$ for every household, at competitive prices and with no transfers:

$$\Omega(x) = \max_{y(e,a)} \int u\big(a(1+r(K)) + e\,w(K) - y(e,a)\big)\,x(\mathrm{d}e,\mathrm{d}a) + \beta\,\Omega(x')$$

$$\text{s.t. } x'(e',A) = \sum_{e} \pi_{ee'}\; x(e,\{a : y(e,a) \in A\}) \;\text{ for every set } A,$$

$$K = \int a\; x(\mathrm{d}e,\mathrm{d}a),$$

  where $x(e,\cdot)$ is the distribution of assets among households with labor e.
- Next period's capital is $K' = \int y(e,a)\,x(\mathrm{d}e,\mathrm{d}a)$, and it sets the prices $r(K')$ and $w(K')$ inside $\Omega(x')$.
- The planner sees what households ignore: its choice of y moves K', and with it next period's prices for everyone.

The planner's first-order condition

- Saving one more unit at (e, a) raises K' by one unit. Next period, a household with (e', a') gains

$$a' \,\mathrm{d}r' + e' \,\mathrm{d}w' = f_{KK}(K', L)\, K' \Big(\frac{a'}{K'} - \frac{e'}{L} \Big) \mathrm{d}K'.$$

- Weighting by marginal utility and adding the private Euler equation:

$$u'(c) \;\geq\; \beta\,(1 + r') \sum_{e'} \pi_{ee'} u'(c') + \Delta'$$

with equality if $a' > \underline{a}$, where $\Delta' = \beta f_{KK}(K', L)\, K' \int u'(c')(a'/K' - e'/L)\, \mathrm{d}x'$.

- This is the two-period Δ with both forces, risk and redistribution. The joint distribution of assets and labor is now the outcome of the history of shocks.

Who are the consumption-poor?

- Δ is an average of $a/K - e/L$, weighted most heavily on the consumption-poor.

- The sign of Δ follows the **factor composition of the income of the consumption-poor**, as a group:
  - if they are mostly poor in labor relative to assets, $e/L < a/K$: $\Delta < 0$, too much capital;
  - if they are mostly poor in assets relative to labor, $a/K < e/L$: $\Delta > 0$, too little capital.

- Which case holds depends on the wealth distribution.

Two calibrations

Two earnings processes

	Unemployment	High wealth dispersion
States of e	$\{0.01,\ 1\}$	$\{1,\ 5.29,\ 46.55\}$
Targets	5% unemployment; 2.6-year spells	earnings Gini 0.60

- Both are annual, with relative risk aversion 2, a capital share of 0.36 and $\delta = 0.08$.

- Each economy is compared across three steady states: first best, market, and constrained optimum.

The unemployment economy has too much capital

	First best	Market	Constrained optimum
Aggregate assets	2.959	3.359	3.279
Interest rate	4.167%	3.219%	3.392%
Gini index for wealth	0.0	0.108	0.105

- The constrained optimum moves capital **down**, toward the first best, and barely changes inequality.

- The consumption-poor are the unemployed, whose labor income is nearly zero: $a/K > e/L$, so $\Delta < 0$.

High wealth dispersion: too little capital

	First best	Market	Constrained optimum
Aggregate assets	1.736	4.017	15.668
Output	1.000	1.353	2.208
Real interest rate	12.740%	4.123%	-2.927%
Gini index for wealth	0.0	0.861	0.864

- Capital rises nearly fourfold and the interest rate turns **negative**; the wealth distribution barely moves.
- The constrained optimum redistributes income, from capital to labor, but hardly redistributes wealth.

Why so much capital?

- Wealth is very skewed and earnings less so: the consumption-poor are **wealth-poor**, with $a/K < e/L$.

- With such unequal wealth, Δ is positive and large.

- All redistribution goes through prices, since the planner cannot transfer goods: more capital lowers the return to being lucky. The rich save more, the poor less.

Different answers, one formula

Economy	Consumption-poor are	Δ	Capital
Unemployment	labor-poor	< 0	too much
High dispersion	wealth-poor	> 0, large	far too little
Aiyagari (1994)	wealth-poor	> 0	too little

- The answer is in the **factor composition of the income of the consumption-poor**.
- For the calibration that best matches US wealth inequality, the market economy saves far too little.
- In the paper's third calibration, Aiyagari (1994), wealth is much less dispersed, but the consumption-poor are still wealth-poor: too little capital, by a small margin.

Scope

Which shocks create the externality?

- Two other shocks, in the two-period model with identical households:
- **Valued leisure.** Capital is still overaccumulated, and households **work too little**: more work lowers the wage, and with it the risky part of income.
- **Expenditure or taste shocks.** They do not enter labor or capital income, so price changes leave risk unchanged: the two-period equilibrium is constrained efficient.
- With more periods, these shocks create wealth dispersion, and with it a reason to change aggregate saving.

Limitations the paper flags

- The large undersaving comes from a calibration in which rare, very high earnings states generate the wealth dispersion.
- If wealth dispersion came from **different patience**, the poor would be poor by choice, and the planner would be more willing to leave them poor.
- The utilitarian objective rests on ex ante equality; other weights would change Δ.
- Market incompleteness is assumed, not derived from private information or limited commitment.

Discussion

Discussion

- In the unemployment economy, would unemployment insurance change the planner's answer?

- *It depends on the income of the consumption-poor.* Transfers to the poor make them save less. A benefit that does not scale with the wage leaves their income less exposed to w, so the effect on Δ depends on how the benefit and the wealth distribution change together.

Discussion

- Is a constrained-inefficient equilibrium an argument for a capital subsidy in practice?

- *Not directly.* The sign of the gap depends on the earnings process, too much capital in one calibration and too little in two, and on why households are poor. Reaching the optimum also takes taxes that depend on each household's history, not a uniform subsidy.

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